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Mathematical Modeling and Computer Simulation of Two Orthogonally Forced Transverse Vibrations in Multisegment Hanging Cantilever

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Abstract—A mathematical model on the basis of energy conservation principle for forced vibrations in a vertically hanging cantilever segmented by variation in its cross-sectional dimensions, configuration, attachments and material changes and excited by transverse orthogonal end concentrated forces is developed and a solution procedure with the help of a computer is presented in this work. The validation of complex algebraic expressions is carried out with simple uniform cantilever computation for two cases, one with a single segment and the other with a number of segments.

Keywords—Multisegment hanging cantilever, Orthogonal transverse vibrations, Flexural stiffness, Fundamental mode shape, Anisotropic elasticity, Undamped response.

1. INTRODUCTION

A hanging cantilever will no longer be a simple uniform cantilever for vibrations, if the flexural stiffness (EI) varies with its length. This may be due to variation in one or more properties like cross-sectional area, shape, orientation with respect to a particular set of axes, material properties or attachments. Change in any of these quantities with length will form a new segment, and thus, the total length of the cantilever will be divided into a number of segments each having constant properties over the segment length. A mathematical model to study the response of such multisegment cantilever hanging in a rotating gas field was needed, but it was available for only a simple and single segment cantilever [1,2]. Direct mathematical derivation on the approach of [1,2] was very tedious and complex and could not be simplified because of the number of variable and integration constants. The mathematical derivation in symbolic form for this multisegments cantilever is presented in this work with sufficient details. In symbolic form, the equations reduced to standard forced vibration equations [2]. Its brief introduction and successful use in a particular case was made in [3], but its generalized and detailed symbolic derivation and computer simulation was not given for want of space. With the help of an electronic computer,

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these equations are solved using the approach of [4] already discussed and modified in [5]. As discussed above, the cantilever is divided into n segments (Figure 1) depending upon variation in its area, configuration, orientation, material properties or attached masses, with numbering from bottom to the top, while global or impressed force axes are in a horizontal plane, such that the X-axis is toward the right, Y-axis towards the viewer, and local z-axis and global Z-axis are collinear and point downward to the bottom end of the cantilever. The global origin is at the top, i.e., fixed end of the cantilever and local origin of a segment is at its own upper end. The material may be anisotropic in the direction perpendicular to its length with angle β between one of its anisotropic plane and the global X-axis. In case of isotropic material, the elastic properties with respect to both planes will be the same with zero orientation to the global set of axes.

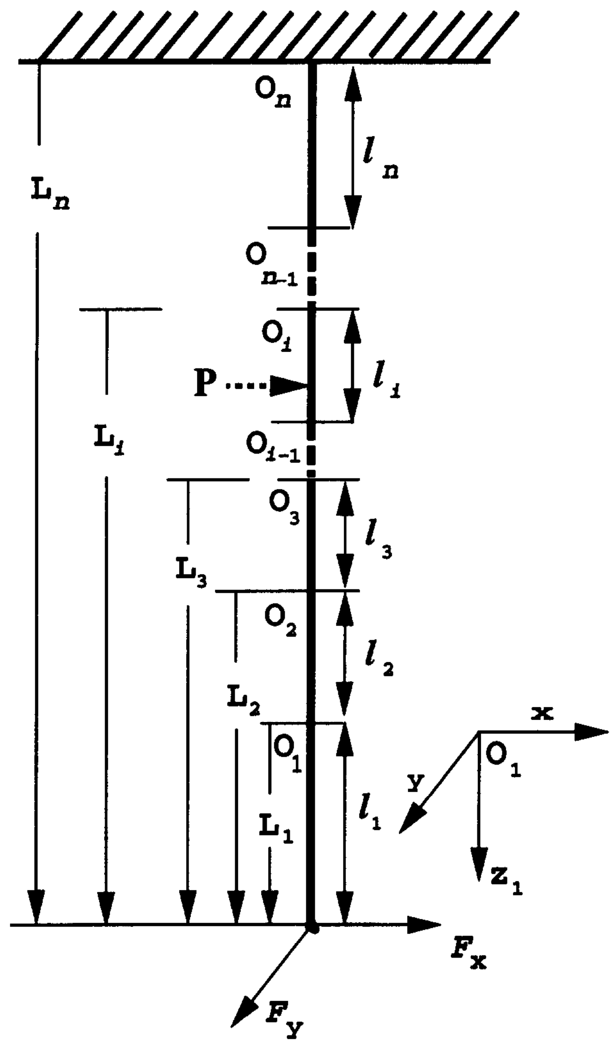

Figure 1. Segment cantilever.

The viscous and mechanical damping of the system are ignored to explicitly see the undamped response of the model and its parameters and the study is restricted only to the fundamental mode of vibration, with mode shape given by deflection of the cantilever under the end-concentrated load. It is also assumed that the simultaneous superposition of forced vibrations in two mutually perpendicular directions of the system do not effect the component quantities of each other. They only effect the net quantities which, in general, are the vectorial sum of the two component quantities like displacements, velocities, accelerations, etc. Also, torsional response of the system to the impressed forces or pure rotational moments and resulting torsional vibrations and displacements are not included in this study due to space limitations and will be addressed in subsequent work. The longitudinal vibrations and lateral effect of weight of suspended masses in the displaced position to bending and mode shape is neglected for simplicity. Also, only the end concentrated exciting forces are considered for the time being and the rest, i.e., intermediate, multipoint, and distributed loads are all deferred for addressing in subsequent work.

2. SEGMENTAL MOMENT OF INERTIA ORIENTATION TO GLOBAL AXIS

The second moment of cross-sectional area about two suitable mutually perpendicular axes may be formulated using a standard procedure given by [6,7]. Then, it can be transformed to moments about the required set of axes using the equations [7],

$$I_{i,x} = I_{i,x_1} \operatorname{Sin}^2 \alpha_i + I_{i,y_1} \operatorname{Cos}^2 \alpha_i - I_{i,x_1 y_1} \operatorname{Sin} 2\alpha_i,$$

and

$$I_{i,y} = I_{i,x_1} \operatorname{Cos}^2 \alpha_i + I_{i,y_1} \operatorname{Sin}^2 \alpha_i + I_{i,x_1 y_1} \operatorname{Sin} 2\alpha_i,$$

where i is segment number, I is the second moment of area or moment of inertia, subscript x and y refer to global set of axes, i.e., orthogonal impressed vibrating forces system of axes, while x_1 and y_1 refer to the local segment set of axes and α_i is the orientation, i.e., angle between the local and global x-axes.

3. ANISOTROPIC ELASTICITY PLANES ORIENTATION TO THE GLOBAL AXIS

For isotropic material, only Young's Modulus E may change from segment to segment with the change of its material. However, if the material is anisotropic with an angle β between one of the anisotropic planes and the global axes of the system, then the component of modulus with respect to global axes are given by

$$E_{i,x} = E_{i,x_2} \operatorname{Cos} \beta_i + E_{i,y_2} \operatorname{Sin} \beta_i,$$

and

$$E_{i,y} = E_{i,y_2} \operatorname{Cos} \beta_i - E_{i,x_2} \operatorname{Sin} \beta_i,$$

where x and y refer to the global set of axes, x_2 and y_2 refer to anisotropic plane directions, and β is the angle between global and anisotropic plane axes.

4. MULTISEGMENT CANTILEVER DEFLECTION

The cantilever deflection is given by the equation

$$y'' = \frac{d^2y}{dz^2} = (l - z)\frac{F}{EI}.$$

Let, in general, the cantilever be of n segments numbered from the bottom to the top and the system of coordinates be local for each segment, such that its origin is at the top of respective segment z in a downward direction along the cantilever length, x and y in the radial direction passing through the center of lever perpendicular to each other and parallel to the global set of X- and Y-axes as shown in Figure 1. Thus, the global X-axis and all local x-axes are coplanar and parallel to the exciting force F_x and the global Y-axis and all local y-axes are coplanar and parallel to the exciting force F_y. Hence, x and y coordinates are same for both local and global set of axes, while z coordinate for the two systems is collinear, but different only in magnitude and the origin. Let l_i be the length of i^{th} segment and L_i be the distance of its origin from bottom end of the lever, i.e., the point of application of the exciting forces F_x and F_y, then,

$$\mathbf{L}_i = l_1 + l_2 + \cdots + l_i.$$

The deflection in y direction of a point P on the i^{th} segment at distance z (i.e., z_i in actual sense) from its segmental origin is given by

$$y_i'' = (L_i - z)\,\frac{F_y}{E_{i,x}I_{i,x}} = A_{i,y}\,(L_i - z)\,F_y$$

and in x direction

$$x_i'' = (L_i - z)\frac{F_x}{E_{i,y}I_{i,y}} = A_{i,x}(L_i - z)F_x,$$

where F_x and F_x are exciting forces and z is taken for z_i, i.e., local z coordinate. Here,

$$A_{i,y} = \frac{1}{E_{i,x}I_{i,x}}$$

and

$$A_{i,x} = \frac{1}{E_{i,y}I_{i,y}}.$$

Dropping the subscripts x and y and considering only the y direction for the time being, the corresponding equations are

$$\begin{aligned} y_i'' &= A_i\,(L_i - z)\,F, \\ y_i' &= A_i\left(L_i z - \frac{z^2}{2}\right)F + b_i, \\ &= \left[A_i\left(L_i z - \frac{z^2}{2}\right) + B_i\right]F, \\ &= F f_o(z,i) \qquad\qquad \text{(say)} \end{aligned}$$

and

$$\begin{aligned} y_i &= A_i\left(\frac{L_i z^2}{2} - \frac{z^3}{6}\right)F + b_i z + c_i, \\ &= \left[A_i\left(\frac{L_i z^2}{2} - \frac{z^3}{6}\right) + B_i z + C_i\right]F, \\ &= F f_1(z,i). \end{aligned}$$

The force divided integration constants B_i and C_i are evaluated from the condition that the slope and displacement of i^{th} segment at $z = 0$ have the same values as that of $(i+1)^{\text{th}}$ segment at $z = l_{i+1}$, i.e., at maximum z. Thus,

$$\begin{aligned} B_i &= \frac{Y_{i+1}'}{F}, \\ C_i &= \frac{Y_{i+1}}{F}. \end{aligned}$$

Here,

$$\begin{aligned} Y_i' &= \max(y_i') = \left[A_i\left(L_i l_i - \frac{l_i^2}{2}\right) + B_i\right]F = F f_o(l,i), \\ Y_i &= \max(y_i) = \left[A_i\left(\frac{L_i l_i^2}{2} - \frac{l_i^3}{6}\right) + B_i l_i + C_i\right]F = F f_1(l,i). \end{aligned}$$

The terminal maximum deflection Y_1 is given by

$$Y_1 = \max(y_1) = \left[A_1\left(\frac{L_1 l_1^2}{2} - \frac{l_1^3}{6}\right) + B_1 l_1 + C_1\right] F = F f_1(l,1) = \frac{F}{q},$$

where

$$q = \frac{1}{f_1(l,1)}.$$

The deflection, slope, and other derived quantities in terms of F and their local variables make the model very complex, which is simplified by normalizing them with Y_1 to have all such quantities in terms of Y_1. Thus, the deflection of i^{th} segment at distance z from top of the segment, in terms of terminal maximum deflection Y_1 is given by

$$\frac{y_i}{Y_1} = \frac{F f_1(z,i)}{F f_1(l,1)} = q f_1(z,i),$$

$$y_i = Y_1 q f_1(z,i).$$

Also,

$$Y_i = Y_1 q f_1(l,i),$$
$$y_i' = Y_1 q f_o(z,i),$$

and

$$Y_i' = Y_1 q f_o(l,i).$$

5. ENERGY EQUATION OF THE SYSTEM

According to energy conservation for an undamped system, the sum of kinetic and potential energy is constant. The simplification of the equation thus arrived by summing up the component potential and kinetic energies of the system at any time, will give the equation of motion of the system. The kinetic and potential energies are evaluated as follows.

5.1. Kinetic Energy of the System

The kinetic energy of the system is given by

$$\mathrm{KE} = \mathrm{KE}_M + \mathrm{KE}_T,$$

where (KE_M) is kinetic energy of the suspended masses and (KE_T) is that of all the segments.

5.1.1. Kinetic energy of the suspended masses

The kinetic energy (KE_{M_i}) of the mass M_i suspended at the lower end of the i^{th} segment is given by

$$\mathrm{KE}_{M_i} = \frac{M_i \dot{Y}_i^2}{2} = \frac{M_i q^2 \dot{Y}_1^2}{2}\left[f_1(l,i)\right]^2 = \frac{M_i q^2 \dot{Y}_1^2}{2} f_2(l,i).$$

The KE_M of all the suspended masses is given by

$$\mathrm{KE}_M = \frac{q^2 \dot{Y}_1^2}{2} \sum_{i=1}^{n} M_i f_2(l,i).$$

M will be zero for the segments with no suspended mass.

5.1.2. Kinetic energy of the segment mass

The KE_T kinetic energy of total n segments is given by

$$\begin{aligned}
\mathrm{KE}_T &= \frac{1}{2}\sum_{i=1}^{n}\int_0^{l_i} m_i\dot{y}_i^2\,dz \\
&= \frac{1}{2}\sum_{i=1}^{n} m_i\int_0^{l_i} \dot{Y}_1^2 q2\,[f_1(z,i)]^2\,dz \\
&= \frac{1}{2}\sum_{i=1}^{n} m_i\dot{Y}_1^2 q^2\int_0^{l_i} [f_1(z,i)]^2\,dz \\
&= \frac{\dot{Y}_1^2 q^2}{2}\sum_{i=1}^{n} m_i f_3(l,i).
\end{aligned}$$

Here, m_i is the mass per unit length of the i^{th} segment and

$$\begin{aligned}
f_3(l,i) &= \int_0^{l_i} [f_1(z,i)]^2\,dz \\
&= \int_0^{l_i}\left[A_i\left(\frac{L_i z^2}{2}-\frac{z^3}{6}\right)+B_i z+C_i\right]^2 dz \\
&= \int_0^{l_i}\left[A_i^2\left(\frac{L_i^2 z^4}{4}+\frac{z^6}{36}-\frac{L_i z^5}{6}\right)+B_i^2 z^2+C_i^2+A_iB_i\left(L_i z^3-\frac{z^4}{3}\right)\right. \\
&\quad \left.+\,2B_iC_i z+A_iC_i\left(L_i z^2-\frac{z^3}{3}\right)\right] dz \\
&= A_i^2\left(\frac{L_i^2 l_i^5}{20}+\frac{l_i^7}{252}-\frac{L_i l_i^6}{36}\right)+\frac{B_i^2 l_i^3}{3}+C_i^2 l_i+A_iB_i\left(\frac{L_i l_i^4}{4}-\frac{l_i^5}{15}\right) \\
&\quad +A_iC_i\left(\frac{L_i l_i^3}{3}-\frac{l_i^4}{12}\right)+B_iC_i l_i^2.
\end{aligned}$$

Thus, the KE of the whole system is given by

$$\mathrm{KE} = \left[\sum_{i=1}^{n}\{M_i f_2(l,i)+m_i f_3(l,i)\}\right]\frac{q^2\dot{Y}_1^2}{2}. \tag{1}$$

5.2. Potential Energy of the System

The potential energy of the system is the sum of the potential energy stored against stiffness of all cantilever segments PE_E, potential energy due to rise of the weight of all segments PE_T, and potential energy due to the rise of suspended masses against gravity PE_M. Thus, the total potential energy PE is given by

$$\mathrm{PE} = \mathrm{PE}_E + \mathrm{PE}_T + \mathrm{PE}_M.$$

The individual potential energies are formulated as follows.

5.2.1. Elastic stiffness energy

The potential energy stored in the hanging cantilever due to its material stiffness is given by

$$\mathrm{PE}_E = \int_0^{Y_1} F\,dY_1.$$

Since $F = qY_1$, therefore,

$$\mathrm{PE}_E = \int_0^{Y_1} qY_1\,dY_1 = \frac{qY_1^2}{2}.$$

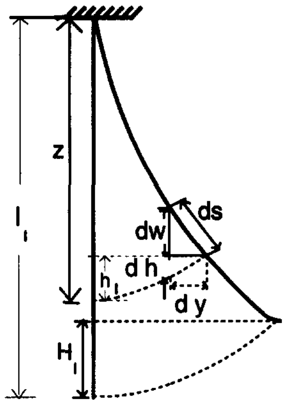

Figure 2. Segment deflection.

5.2.2. Rise of an element against gravity

As shown in Figure 2, the rise of an element ds at a distance z from the top of the segment against gravity during bending is given by

$$
\begin{aligned}
h &= \int_0^z dh = \int_0^z (ds - dw) \\
&= \int_0^z \left(ds - \sqrt{ds^2 - dy^2}\right) \\
&= \int_0^z \left[1 - \left\{1 - (y')^2\right\}^{1/2}\right] ds \\
&= \int_0^z \left[1 - \left\{1 - \frac{1}{2}(y')^2\right\}\right] ds \\
&= \frac{1}{2}\int_0^z (y')^2\, ds,
\end{aligned}
$$

since $y' \ll 1$. Thus, for i^{th} segment,

$$
\begin{aligned}
h_i &= \frac{1}{2}\int_0^z (y_i')^2\, ds \\
&= \frac{1}{2}\int_0^z \left[A_i\left(L_i s - \frac{s^2}{2}\right) + B_i\right]^2 F^2\, ds \\
&= \frac{Y_1^2 q^2}{2}\left[A_i^2\left(\frac{L_i^2 z^3}{3} + \frac{z^5}{20} - \frac{L_i z^4}{4}\right) + A_i B_i\left(L_i z^2 - \frac{z^3}{3}\right) + B_i^2 z\right] = \frac{Y_1^2 q^2}{2} f_4(z,i),
\end{aligned}
$$

and its maximum value is given by

$$H_i = \frac{q^2 Y_1^2}{2} f_4(l,i).$$

The overall or global rise of an element is given by

$$u_i = h_i + \sum_{j=i+1}^{n} H_j.$$

The overall or global rise of a segment bottom end is given by

$$U_i = \sum_{j=i}^{n} H_j.$$

5.2.3. Potential energy of the suspended masses

The potential energy of a suspended mass M_i is given by

$$\begin{aligned}\mathrm{PE}_{M_i} &= M_i g U_i = M_i g \sum_{j=i}^{n} H_j \\ &= \frac{M_i g Y_1^2 q^2}{2} \sum_{j=i}^{n} f_4(l,j).\end{aligned}$$

The potential energy of all the suspended masses is given by

$$\mathrm{PE}_M = \sum_{i=1}^{n} M_i g U_i = \frac{g Y_1^2 q^2}{2} \sum_{i=1}^{n} \left[M_i \sum_{j=i}^{n} f_4(l,j) \right].$$

5.2.4. Potential energy due to segments weight

The potential energy due to weight of the i^{th} segment is given by

$$d(\mathrm{PE}_T)_i = m_i g u_i \, dz,$$

where m_i is mass per unit length of i^{th} segment of cantilever. Thus,

$$\begin{aligned}(\mathrm{PE}_T)_i &= m_i g \int_0^{l_i} u_i \, dz \\ &= m_i g \int_0^{l_i} \left(h_i + \sum_{j=i+1}^{n} H_j \right) dz \\ &= \frac{m_i g Y_1^2 q^2}{2} \left[\left\{ A_i^2 \left(\frac{L_i^2 l_i^4}{12} + \frac{l_i^6}{120} - \frac{L_i l_i^5}{20} \right) + \frac{B_i^2 l_i^2}{2} + A_i B_i \left(\frac{L_i l_i^3}{3} - \frac{l_i^4}{12} \right) \right\} + l_i \sum_{j=i+1}^{n} f_4(l,j) \right] \\ &= \frac{m_i g Y_1^2 q^2}{2} \left[f_6(l,i) + l_i \sum_{j=i+1}^{n} f_4(l,j) \right] = \frac{m_i g Y_1^2 q^2}{2} f_5(l,i).\end{aligned}$$

PE_T for all segments is given by

$$\begin{aligned}\mathrm{PE}_T &= \sum_{i=1}^{n} (\mathrm{PE}_T)_i \\ &= \frac{Y_1^2 q^2 g}{2} \sum_{i=1}^{n} m_i f_5(l,i).\end{aligned}$$

Thus, the total PE of the system is given by summing up its components as

$$\mathrm{PE} = \frac{q {Y_1}^2}{2} + \frac{g Y_1^2 q^2}{2} \sum_{i=1}^{n} \left[M_i \sum_{j=i}^{n} f_4(l,j) \right] + \frac{Y_1^2 q^2 g}{2} \sum_{i=1}^{n} m_i f_5(l,i). \tag{2}$$

6. EQUATION OF MOTION OF THE SYSTEM

The energy conservation equation of the system is

$$\mathrm{KE} + \mathrm{PE} = G, \qquad \text{where } G \text{ is a constant.}$$

With the values of KE and PE from (1) and (2), the above equation becomes

$$\left[q^2 \sum_{i=1}^{n} \{M_i f_2(l,i) + m_i f_3(l,i)\}\right] \frac{\dot{Y}_1^2}{2} + \left[q + gq^2 \sum_{i=1}^{n} \left\{ M_i \sum_{j=i}^{n} f_4(l,j) + m_i f_5(l,i) \right\}\right] \frac{Y_1^2}{2} = G.$$

Differentiating with respect to time, simplifying, rewriting the subscript y, and replacing Y_1 by simply Y, the equation becomes

$$\left[\sum_{i=1}^{n} \{M_i f_2(l,i)_y + m_i f_3(l,i)_y\}\right] \ddot{Y} + \left[\frac{1}{q_y} + g \sum_{i=1}^{n} \left\{ M_i \sum_{j=i}^{n} f_4(l,j)_y + m_i f_5(l,i)_y \right\}\right] Y = 0,$$

or

$$M_y \ddot{Y} + K_y Y = 0. \tag{3}$$

Similarly, the equation for x direction is

$$M_x \ddot{X} + K_x X = 0, \tag{4}$$

where

$$M_y = \left[\sum_{i=1}^{n} \{M_i f_2(l,i)_y + m_i f_3(l,i)_y\}\right],$$

$$M_x = \left[\sum_{i=1}^{n} \{M_i f_2(l,i)_x + m_i f_3(l,i)_x\}\right],$$

$$K_y = \left[\frac{1}{q_y} + g \sum_{i=1}^{n} \left\{ M_i \sum_{j=i}^{n} f_4(l,j)_y + m_i f_5(l,i)_y \right\}\right],$$

and

$$K_x = \left[\frac{1}{q_x} + g \sum_{i=1}^{n} \left\{ M_i \sum_{j=i}^{n} f_4(l,j)_x + m_i f_5(l,i)_x \right\}\right].$$

If F_x and F_y are exciting forces in global X and Y direction acting at the tip of the cantilever, then their respective equations of motions are given by

$$M_y \ddot{Y} + K_y Y = F_y, \tag{5}$$

and

$$M_x \ddot{X} + K_x X = F_x. \tag{6}$$

Here, viscous and mechanical damping of the system is ignored to clearly see the undamped response of the model and its parameters.

7. SYSTEM DATA AND COMPUTATIONAL TECHNIQUES

From the preceding sections, the mathematical model, therefore, is

$$M_y\ddot{Y} = F_y - K_y Y \tag{7}$$

and

$$M_x\ddot{X} = F_x - K_x X, \tag{8}$$

where F_x and F_y are random exciting forces. The computational model is developed from this using iterative computational technique [4] for initial value problems for special second-order differential equations. Automatic halving of the time step in the case of nonconvergence and intelligently doubling it after few time steps to get a faster solution with corresponding changes of related terms in two cases have, however, been added to the original scheme. A set of data of a simple physical system in a consistent system of units is used for this study. The displacements x and y of the tip of cantilever at any time are given by the solution of the above equations and the net displacement of the tip from the center or mean position is given by

$$r = \sqrt{x^2 + y^2}, \tag{9}$$

while the angle of this displacement with the x-axis is given by

$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right). \tag{10}$$

The data of a particular physical uniform cantilever of a uniform material excited transversely by an orthogonal pair of varying forces impressed by a rotating gas is studied, once taking it as single segment and once as multisegments. Out of the results plotted in Figures 3 and 4, the two orthogonal forces are normalized and the Y-component of displacement is displaced from its actual position for the purpose of a clear view in the comparative study.

8. RESULTS AND DISCUSSIONS

The model for a simple cantilever, i.e., a single segment uniform cantilever is valid as quoted in [1] and [2]. The multisegment cantilever model is validated by comparison of the coefficients of $\ddot{Y}$, Y, $\ddot{X}$, and X in equations (3) and (4), once evaluating as single and once as multisegment in one and the same cantilever without any change in configuration of cross-section or modulus of elasticity. The two have been found exactly the same in both cases along with the natural frequencies. The complete performance of this computational model with two orthogonal forces in the two cases for a simple cantilever is shown in Figures 3 and 4. Response of the system with same exciting forces computed as single segment (Figure 3) and that of the same system divided into eight segments of different lengths (Figure 4), show their complete and exact similarity. This confirms the validity of the model as far as uniform cantilever of one and the same material is concerned. The multisegments model of different cross-sectional configuration, however, needed validation using experimental measurements of the variables for the practical system and one such particular system is validated in [3].

9. CONCLUSION

This mathematical model and solution procedure can be used for study of response of different parameters of a physical segmented hanging cantilever subjected to forced vibrations excited by orthogonal transverse forces. Based on this study, either some suitable parameters can be adjusted or some additional measures can be taken to avoid vibration catastrophe. This has opened the door for premanufacture anticipation of the response of any physical segmented hanging cantilever and that of its suggested safety measures, and hence, may successfully help in control of vibrational devastation.

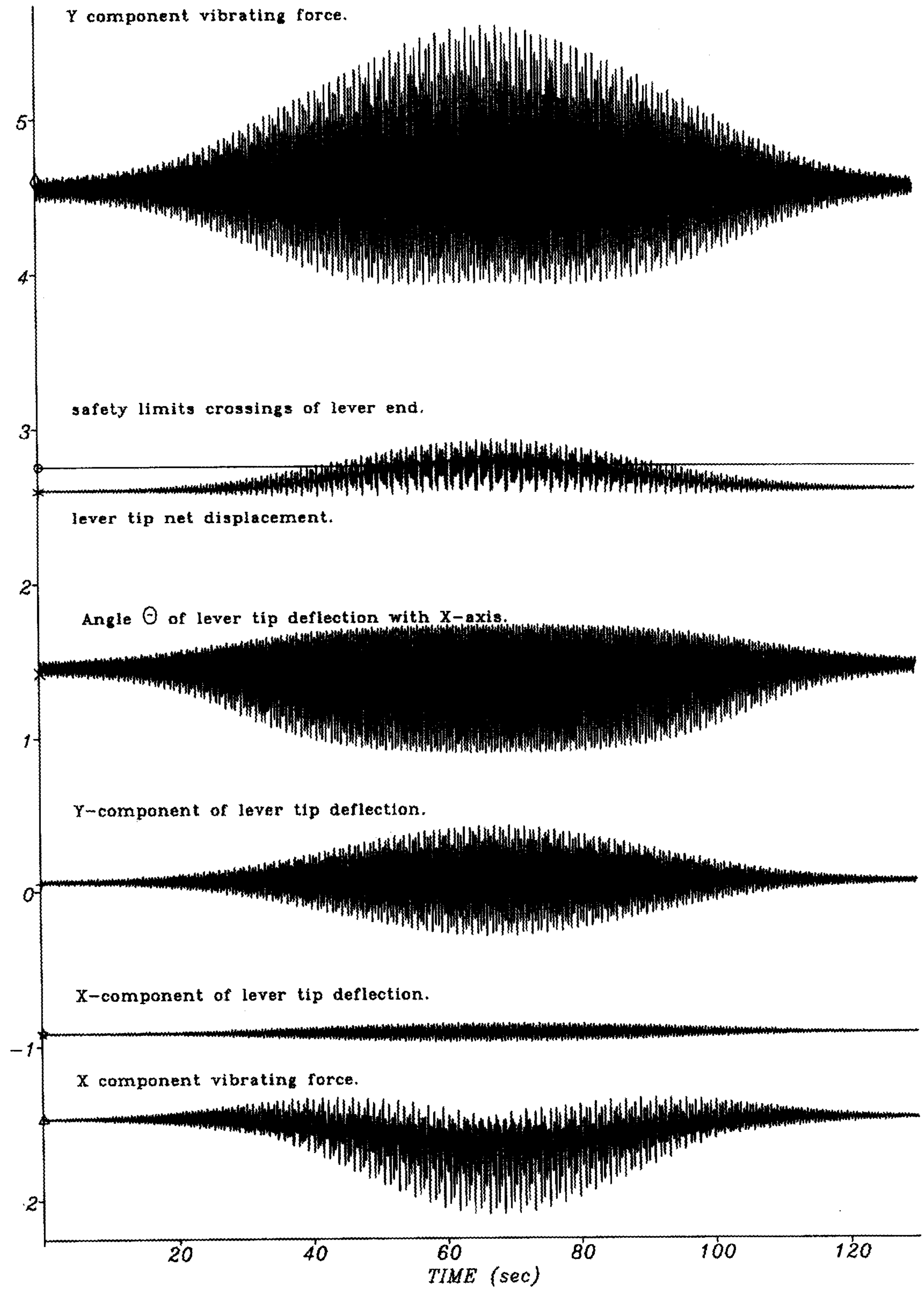

Figure 3. X and Y component vibrating forces and the resulting tip displacements along with its throw beyond limit plotted vs. time for single segment lever activated by rotating gas.

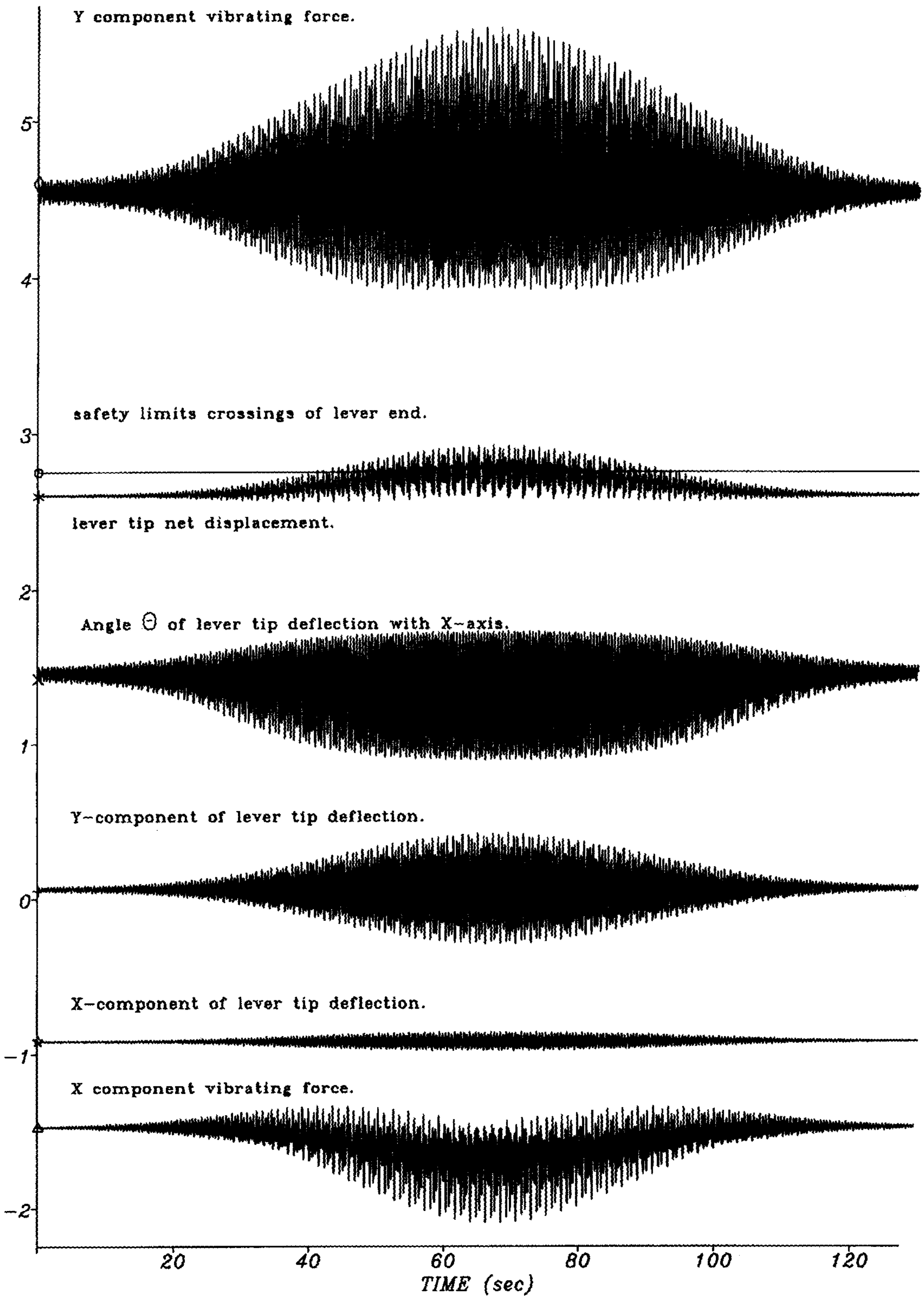

Figure 4. X and Y component vibrating forces and the resulting tip displacements along with its throw beyond limited plotted vs. time for multisegment (i.e., 8) segment lever activated by rotating gas.

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